On the XOR of Multiple Random Permutations

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Introduction

PRP PRF

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$$\begin{tikzcd} \text{PRP} & & \text{PRF} \arrow[ll, bend right, "\text{Luby-Rackoff / Feistel}"'] \end{tikzcd}$$

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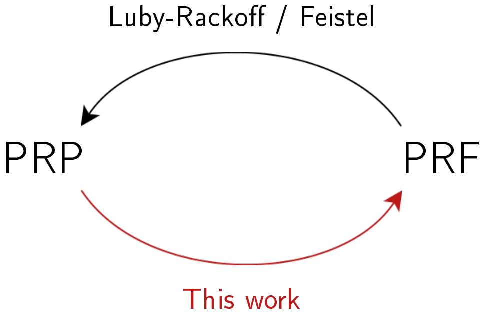

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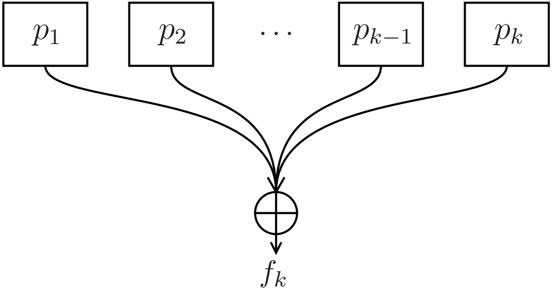
- All: secure PRFs up to birthday bound
- XOR of multiple PRPs: $E_{K_1}(x) \oplus \cdots \oplus E_{K_k}(x)$?

XOR of Multiple Permutations



$$f_k(x) = p_1(x) \oplus \cdots \oplus p_k(x)$$

Instantiations

Secret Permutations

- Based on $E_{K_1}, \ldots, E_{K_k}$
- Adversary can only evaluate f_k

  $\longrightarrow$ indistinguishability

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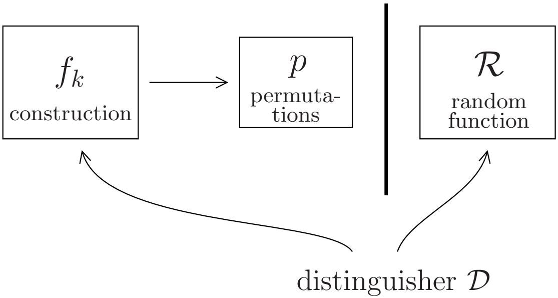
Public Permutations

- Based on stand-alone $p_1, \ldots, p_k$
- Adversary can evaluate f_k and $p_1, \ldots, p_k$

$\longrightarrow$ indifferentiability

Indistinguishability of f_k (p_i secret)

Indistinguishability of f_k: Security Model



- $p = (p_1, \ldots, p_k)$ random n-bit permutations
- $\mathcal{R}$ random n-bit function
- Distinguisher $\mathcal{D}$ computationally unbounded

Indistinguishability of f_k: State of the Art

indistinguishability (p_i secret)	k	bound	reference
	≥ 1	$2^{\frac{k}{k+1}n}$	[Lucks00]
	2	$2^n/n^{2/3}$	[Bellare99]
	2	2^n	[Patarin08]
	≥ 3	$2^{\frac{2k+1}{2k+2}n}$	[CogliatiLP14]

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Conjectured 2^n

Indistinguishability of f_k: Short Proof

Theorem For all $k \geq 2$, we have $\mathsf{Adv}^{\mathrm{dist}}_{f_{k+1}}(\mathcal{D}) \leq \mathsf{Adv}^{\mathrm{dist}}_{f_k}(\mathcal{D})$

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- Security of f_{k+1}
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Lemma [Patarin08] We have $\mathsf{Adv}^{\mathrm{dist}}_{f_2}(\mathcal{D}) = \mathcal{O}(q/2^n)$

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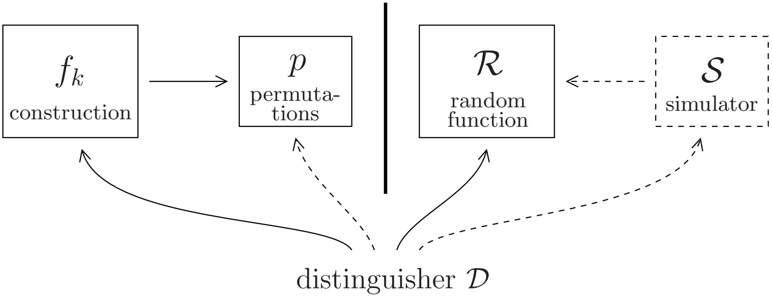
Lemma [Patarin08] We have $\mathsf{Adv}^{\mathrm{dist}}_{f_2}(\mathcal{D}) = \mathcal{O}(q/2^n)$

Corollary For all $k \geq 2$, we have $\mathsf{Adv}^{\mathrm{dist}}_{f_k}(\mathcal{D}) = \mathcal{O}(q/2^n)$

Indifferentiability of f_k (p_i public)

Indifferentiability of f_k: Security Model



- Extends indistinguishability: structure of f_k is known
- f_k indifferentiable from $\mathcal{R}$ if $\exists$ simulator $\mathcal{S}$ such that (f_k, p) and $(\mathcal{R}, \mathcal{S})$ are indistinguishable

Indifferentiability of f_k: State of the Art

indifferentiability	k	bound	reference
(p_i public)	2	$2^{n/2}$	[MandalPN10]
	2	$2^{2n/3}$	[MandalPN10]

Our Contribution

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- Flaw in proof of [MandalPN10]
- Re-confirmation and generalization of bound

Indifferentiability of f_k: New Result

Theorem For all $k \geq 2$, there exists a simulator $\mathcal{S}$ such that

$$\mathsf{Adv}^{\mathrm{diff}}_{f_k,\mathcal{S}}(\mathcal{D}) \leq \frac{4q^3}{2^{2n}} + \frac{3n^{1/2}q^{3/2}}{2^n} + \frac{2}{2^n}$$

- Old bound: $\frac{96q^3}{2^{2n}} + \frac{1}{2^{11n}}$ [MandalPN10]

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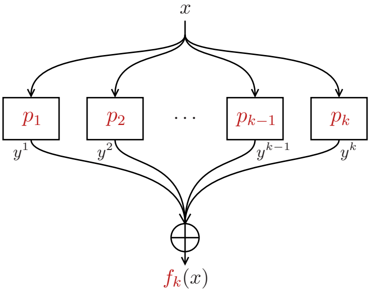
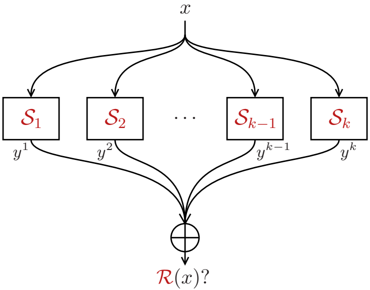
- Simulator $\mathcal{S}$ and proof similar to the old ones
- Now: high-level intuition

Indifferentiability of f_k: Simulator





Goal of Simulator

- Tries to answer queries such that $(f_k, p) \approx (\mathcal{R}, \mathcal{S})$

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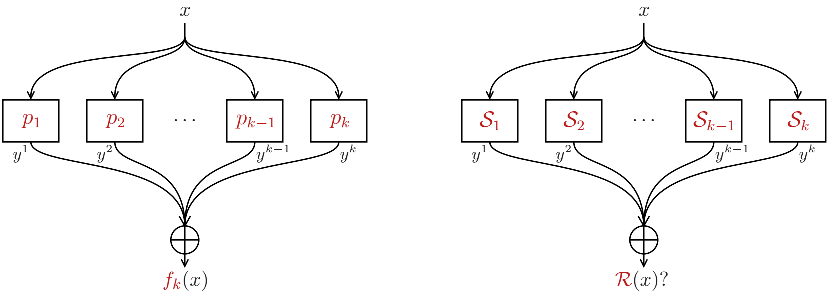

Goal of Simulator

- Tries to answer queries such that $(f_k, p) \approx (\mathcal{R}, \mathcal{S})$
- Query-responses $(x, y^1, \ldots, y^k)$ should satisfy
  - $\mathcal{R}(x) = y^1 \oplus \cdots \oplus y^k$
  - x and y^ℓ permutation-wise distinct for all $\ell = 1, \ldots, k$

Indifferentiability of f_k: Proof Idea

Patarin's H-coefficient Technique

- Each conversation defines a transcript
- Define good and bad transcripts

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↰ prob. ratio for good transcripts

- Trade-off: define bad transcripts smartly!

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- Transcript is bad if $|N(z)| \geq \frac{24q^2}{2^n - q}$ for some z

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- Attacker can assure $|N(z)| \geq q/2$ trivially
  $\longrightarrow \mathbf{P}\,(\text{bad}) = 1$

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  - Description of simulator thwarted to $k = 2$

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Thank you for your attention!

SUPPORTING SLIDES

Indifferentiability of f_k: Simulator

Forward Query $\mathcal{S}(x)$

1. Generate random $y^3, \ldots, y^k$ permutation-wise
2. Query $\mathcal{R}(x)$
3. Generate random y^1, y^2 permutation-wise such that

$$y^1 \oplus y^2 = \mathcal{R}(x) \oplus y^3 \oplus \cdots \oplus y^k$$

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Inverse Query $\mathcal{S}_\ell^{-1}(y^\ell)$ (now: $\ell = 1$)

1. Generate random $y^2, \ldots, y^{k-1}$ permutation-wise
2. Generate random x permutation-wise and query $\mathcal{R}(x)$
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3. Set $y^k = \mathcal{R}(x) \oplus y^1 \oplus \cdots \oplus y^{k-1}$
4. If y^k collides with old value: return to 2.